COMAP

INTERNATIONAL MATHEMATICAL MODELING CHALLENGE

2021

CONTEST RESULTS AND PAPER

IM²C promotes the teaching of mathematical modeling and applications at all educational levels for all students. It is based on the firm belief that students and teachers need to experience the power of mathematics to help better understand, analyze, and solve real world problems outside of mathematics itself –and to do so in realistic contexts. The Challenge has been established in the spirit of promoting educational change.

www.immchallenge.org

2021 IM^{2}C

The 7th annual International Mathematical Modeling Challenge (IM^{2}C) culminated with two Outstanding Teams. Congratulations to these teams and all the teams that participated in the 2021 IM^{2}C. This year, due to the continued effects of Covid-19, there was no formal in-person IM^{2}C awards ceremony. Rather, ICME Shanghai, China, acknowledged teams virtually at a hybrid conference event on July 15, 2021. This event featured videos prepared by the winning teams. IM^{2}C has made resources available to schools and countries/regions of the top teams to fund local ceremonies scheduled as their situations permit.

The IM^{2}C continues to be a rewarding experience for students, advisors, schools, and judges. A total of 51 teams, with up to 4 students each, representing 27 countries/regions competed in this year's international round.

The purpose of the IM^{2}C is to promote the teaching of mathematical modeling and applications at all educational levels for all students. It is based on the firm belief that students and teachers need to experience the underlying power of mathematics to help better understand, analyze, and solve real world problems outside of mathematics itself—and to do so in realistic contexts. The Challenge has been established in the spirit of promoting educational change.

For many years there has been an increased recognition of the importance of mathematical modeling from universities, government, and industry. Modeling courses have proliferated in undergraduate and graduate departments of mathematical sciences worldwide. Several university modeling competitions are flourishing. Yet at the school level, even amid signs of the growing recognition of modeling's centrality, there are only a few such competitions with many fewer students participating. One important way to influence secondary school culture, and teaching and learning practices, is to offer a high-level prestigious secondary-school contest that has both national and international recognition. With this in mind, we founded the International Mathematical Modeling Challenge (IM^{2}C) in 2014 and launched the 1st annual Challenge in 2015.

The IM^{2}C is a true team competition held over a number of days, with students able to use any inanimate resources. Real problems require a mix of different kinds of mathematics for their analysis and solution. And, real problems take time and teamwork. The IM^{2}C provides students with a deeper experience of how mathematics can explain our world, and the satisfaction of applying mathematics to a real world problem to develop a model and solution.

Plans for 2022

We invite countries to enter up to two teams, each with up to four students and one teacher/faculty advisor. The contest will begin in March and end in May. During that timeframe, teams will choose five (5) consecutive days to work together on the problem. The faculty advisor must then submit the paper and certify that students followed the contest rules.

The International Expert Panel will judge the papers in early June and will announce winners by late June. Papers will be designated as Outstanding, Meritorious, Honorable Mention, and Successful Participant with appropriate plaques and certificates given in the name of students, their advisor, and their schools.

Plans for the 2022 awards are still being finalized. Complete information about IM^{2}C is at www.immchallenge.org

The IM^{2}C International Organizing Committee

Solomon Garfunkel, *COMAP, USA – Chair*

Keng Cheng Ang, *National Institute of Education, Singapore*

JunFeng Yin, *Tongji University, China*

Alfred Cheung, *NeoUnion ESC Organization, China Hong Kong (SAR)*

Frederick Leung, *University of Hong Kong, China Hong Kong (SAR)*

Vladimir Dubrovsky, *Moscow State University, Russia*

Henk van der Kooij, *Freudenthal Institute, The Netherlands*

Mogens Allan Niss, *Roskilde University, Denmark*

Ross Turner, *Australian Council for Educational Research, Australia*

Jie "Jed" Wang, *University of Massachusetts, USA*

The 2021 IM^{2}C Problem: Who is the Greatest? Maradona or Pelé? Biles or Khorkina?

We read all the time in the sports pages about an athlete being called the **G.O.A.T.**—the *Greatest Of All Time*. What does that really mean and how can that truly be determined?

For the purpose of this IM^{2}C problem, we consider two types of "**sports**"—and, we allow "sports" to be defined broadly.

IM^{2}C Funding

Funding for planning and organizational activities is provided by IM^{2}C co-founders and co-sponsors: *Consortium for Mathematics and its Applications* (COMAP), a not-for-profit company dedicated to the improvement of mathematics education, and *NeoUnion ESC Organization* in China Hong Kong (SAR).

1. **Individual Sports.** An individual person competes against one or more other players either **"one-on-one"** or against an inanimate standard (highest or lowest ranking or score or time measure). Competitors do not necessarily physically interact with each other (e.g. golf, marathon running, swimming, chess, and table tennis), but they could interact physically with their opponent (e.g. boxing, wrestling). Winners in these sports competitions are individuals and not teams.

2. **Team Sports.** A group of individuals competes against another group of individuals and the competition includes multiple interactions between any and all players physically, strategically, or through equipment. Examples of team sports include basketball, hockey, American football, International football (soccer), and water polo. Winners in team sports competitions are teams. Although team sports incorporate a variety of position players, individuals, like Michael Jordan in basketball, may stand out.

   How is the *greatest* determined? Sometimes we judge sports figures based on an accumulation of records and results over several years such as Palmer or Nicholas in golf, Bonnie Blair in women's speed skating, Ma Long in table tennis, or Tom Brady in American football. Sometimes we call them great because of one athletic feat such as Bob Beamon's world record long jump in the 1968 Olympics or Nadia Comaneci's perfect 10s in gymnastics during the 1976 Olympics. Some athletes come up at a time of rich competition or establish a famous rivalry like Evert and Navratilova in women's tennis or Ali and Frasier in boxing.

   Top Sport, a sports network, has hired your team to consider models to measure "greatness." Your first assignment is to develop a model for individual sports and use your model to determine the "G.O.A.T." of one individual sport.

   Given that there may be different divisions in sports having restrictions for membership, such as separate competitions for men and women, or various weight classes in boxing, wrestling, and weightlifting, you should consider each of these divisions a single sport (e.g. women's gymnastics, men's gymnastics, featherweight boxing, lightweight boxing) each able to have its own G.O.A.T.

Requirements

1. **Warm Up.** Consider the individual sport of singles women's tennis over the period of one year in 2018. The best women's tennis players play in the four Grand Slam tournaments (Australian Open, French Open, Wimbledon, and US Open). We provide results of these tournaments on pages 4-6 (<u>2018 Grand Slam Results</u>). Who among these players was the greatest?

   a. Develop a mathematical model for determining the greatest woman tennis player in 2018 on the basis of Grand Slam tournament results. Discuss your choice of factors/variables and the development of your model.

   b. Use your model to choose the greatest woman tennis player of 2018. Analyze your result.

2. **Finding the G.O.A.T. of any Individual Sport.** Note that in #1 you only looked at one year of a particular individual sport. Now, consider determining the *Greatest Of <u>All Time</u>* (G.O.A.T.) of <u>ANY</u> individual sport.

   a. Choose one example of a individual sport (other than Women's Tennis), and develop a mathematical model (or models) from any factors and data you find significant, measurable, and obtainable for determining the G.O.A.T. in that sport. Analyze your result. Ensure you document any resources used in gathering data and information about your sport.

   b. Discuss any changes your G.O.A.T. model from #2.a. would require to determine the G.O.A.T. of any individual sport. You DO NOT need to develop a new model, but address and explain how models for other individual sports would differ.

3. **Extending Your G.O.A.T Model from #2.** Now think about team sports. Discuss any changes your G.O.A.T. models from #2 would require to determine the G.O.A.T. of a team sport. You DO NOT need to develop a new model, but address and explain how a model for team sports would differ from your models for individual sports.

4. **Letter.** Write a one-page letter to the Director of *Top Sport* describing your team's model and your example of the G.O.A.T. for your selected individual sport. The Director is an executive who understands little about math modeling and science, but is interested in general principles and your key findings.

   Note that IM^{2}C is aware of available resources and references that address and discuss this question. It is not sufficient to simply re-present any of these models or discussions, even if properly cited. Any successful paper MUST include development and analysis of your own team's model and clearly explaining the difference between your model and any referenced existing ranking system.

   To view the complete problem, go to http://immchallenge.org/Contests/2021/2021_IMMC_Problem.pdf

The 2021 IM^{2}C International Judges' Commentary

Angeles Dominguez Cuenca
Frank Giordano
Ruud Stolwijk

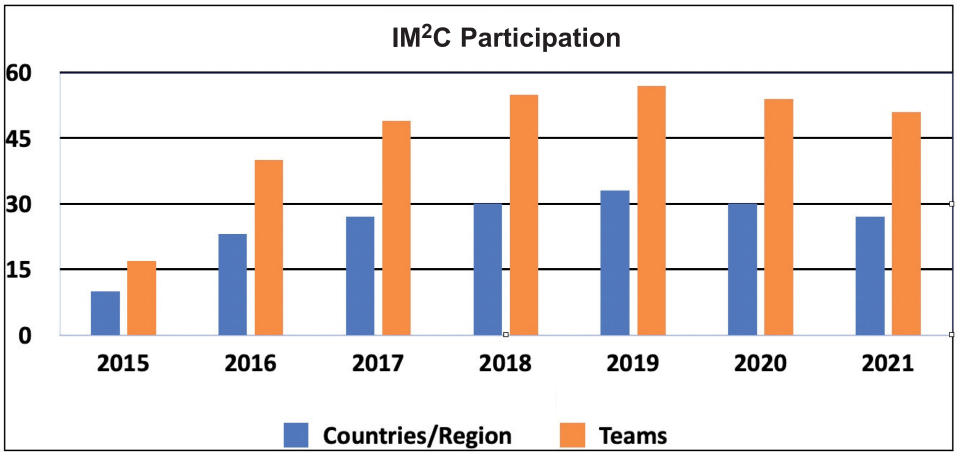
Every year, the IM^{2}C judges are impressed by the students' creativity, team collaboration, time invested, and mathematical knowledge as demonstrated in their Challenge solutions. This year was no exception as students considered finding a model for the Greatest of All Time. We value and recognize the quality and originality of the solution papers. Student teams worked to interpret the problem, identify and relate the variables, research for deeper understanding of the scenario, build their model, and finally draft a coherent and fluent report of their solution.

In 2021, teams in the international round represented 27 countries/regions from a total of 51 schools. The Expert Panel of Judges recognizes that each country/region selects two teams at most, meaning that the number of teams who participated in the 2021 Challenge is much greater. Having that many teachers, parents, schools, and country/region Challenge coordinators motivating and energizing so many student teams to solve the problem is very exciting. We are thrilled by the commitment shown by all. We encourage these mathematics community members to share their experiences with others and to continue motivating, inspiring, and supporting student participation. Students who experience the initial struggle of the Challenge, and then enjoy the success of completing their solution, benefit personally and mathematically. We congratulate all teams who took part in IM^{2}C 2021.

This year's Challenge was to propose a mathematical model to determine the athlete considered the best ever to compete, perform, or participate in a specific sport or activity, called the *G.O.A.T—the Greatest of All Time.* Finding the greatest competitor across all time in any sport is certainly a difficult task. Developing a model to do so is quite a challenge.

A comprehensive IM^{2}C report requires a summary sheet, an introduction with a problem restatement, a presentation and discussion of the mathematical modeling processes used in developing the model(s) (assumptions with justifications, factors or variables defined, a final mathematical model), application of the model to the problem, and analysis of the results to include strengths and weaknesses, a

School, Location	Advisor	Team Members
Diocesan Girls' School **China Hong Kong (SAR)**	YEUNG Po Ki Dora	CHAN Tsz Ching CHEUNG Chun Yan Alicia IP Tsz Oi TSUI Vivienne
Hwa Chong Institution **Singapore**	NG TONG CHEONG	JIANG ZHIHENG TAN YI KAI YU WENHAO TAN JIECONG



Numbers of Participating Countries/Regions and Teams 2015-2021

The 2021 IM^{2}C Expert Panel

Frank Giordano,
Naval Post Graduate School, USA – Chair

Konstantin K. Avilov,
Institute for Numerical Mathematics, Russia

Ruud Stolwijk,
Cito, The Netherlands

Liqiang Lu,
Fudan University, China

Jill Brown,
Australian Catholic University, Australia

Yang Wang,
The Hong Kong University of Science and Technology, China Hong Kong (SAR)

Dra. Ángeles Domínguez Cuenca,
Tecnológico de Monterrey, Mexico

sensitivity analysis, and conclusions. Teams must address any questions and requirements found within the problem statement and, in many cases, write some sort of letter or memorandum. For the 2021 Challenge, the following paragraphs more specifically discuss each of these parts of a report.

Summary Sheet. The summary is a very important part of the IM^{2}C report. The summary sheet is the first thing a judge reads. It provides the first chance for a team to tell the judge about their processes and results. A summary should clearly describe the approach to the problem and the most relevant conclusions. The summary should invite and motivate the reader to want to know more about the team's solution. Judges place considerable weight on the summary, especially in the first round of judging when the first 'cut' is made. For the 2021 Challenge, the summary sheet should have described the team's modeling processes, stated their 2018 Women's Tennis G.O.A.T., indicated the individual sport they chose and stated their determined G.O.A.T. for that sport, and discussed a few differences they found between their individual and team sport models.

Introduction and Problem Restatement. Teams usually begin with a Table of Contents followed by an introduction presenting additional information about the problem topic and general scenario. Teams then restate the problem in their own words identifying the specific requirements they intend to address. The 2021 Challenge required students to develop several G.O.A.T. models to address variations in the problem requirements. Judges use this part of the report as a preview and overview as to how the team approached the problem.

Mathematical Modeling Processes. The most important part of a team's submission is their model. All other parts of their paper support the development, use, and analysis of the model. Teams must explain the mathematics used in a logical and clear manner. This is the heart of modeling! Make sure as you model you always stay in touch with reality. Since the problem is a real-life problem, it is quite essential that teams reflect and critically judge their mathematical solution as calculated.

In order to begin each modeling requirement, teams determined any assumptions required for their model. Assumptions help to simplify the problem or to limit it to conditions for which teams can find a solution. Too many assumptions may lead to oversimplification. Every assumption should relate specifically to the developed model and teams should justify their assumptions by indicating that relationship.

As students started on the G.O.A.T. Challenge, they addressed the warm-up requirement of considering only the individual sport of singles women's tennis during one year using data from the four 2018 Grand Slam tournaments (Australian Open, French Open, Wimbledon, and US Open). Most teams began by presenting some information about women's tennis and the tournaments, and describing the available data. To develop a mathematical model, teams identified and defined appropriate factors and variables, and determined the relationships among these factors and variables. Teams recognized that the best player was not simply the person with the greatest number of match, set, or game wins, or even the player reaching the most finals or semi-finals of a tournament. For example, a tennis match with set scores of 7-6, 0-6, 7-6 results in more game wins for the losing player than the winning player. This shows that testing your model in all situations is crucial. Although many factors play a role and winning is important, better teams incorporated the relationship between the rankings of a player and her opponent in terms of difficulty of a match, as well as how dominant a player was (margin of win) against these opponents. Some teams considered predicting results of matches between players who did not play against each other to make a comparison. Once developed, teams had to use and test their model to determine the 2018 Women's Tennis G.O.A.T and justify their result. Better teams analyzed their model by comparing the model to the actual 2018 women's tennis world rankings and discussing why their result was either similar or different. Teams discussed the strengths and limitations of their model. Additionally, team's analyzed their model by doing sensitivity analysis and error analysis with respect to the weightings or coefficients they used in the model.

The Challenge next asked teams to find the G.O.A.T. of any individual sport (as defined in the problem statement) of their choosing. Generally, judges preferred teams to choose a sport other than tennis to show the more general modeling ability of the team. Teams chose a variety of sports including badminton, chess, swimming, boxing, speed skating, artistic gymnastics, and darts. It appeared students truly enjoyed selecting their own sport. As in the warm-up, teams presented information about their

USA Participation

In the USA, we invite all teams that successfully compete in the HiMCM contest and are awarded a designation of Meritorious or above (Meritorious, Finalist, or Outstanding) to compete in the IM^{2}C. From these participants, U.S. Judges select the two top teams to move on and represent the USA in the IM^{2}C international round. To participate in HiMCM in November 2021, visit www.comap.com.

sport and its rules and competitions. They selected and defined factors and variables, and developed their mathematical model. Judges expected the models for this requirement to differ as now teams had to look across all participants over all years for their sport (versus just one year of women's tennis). This resulted in teams making additional and different assumptions. For example, limiting the candidates to those who had won at least one world championship or who had set a sport record or who had played the sport for a certain number of years at the highest level. Teams needed to determine the data they would consider. Would they use all available data or only international competition data such as world championships or the Olympics? In looking for the greatest player across all time, teams considered whether and how they should account for changes in equipment or rules over the years. Once developed, teams used their model to choose an actual G.O.A.T., and analyzed and justified their results. They discussed strengths and limitations, as well as conducted sensitivity and error analysis to test variable choices in their model.

Teams then extended their particular sport model to discuss changes required in order to determine the G.O.A.T. of any individual sport. The better teams discussed and addressed the differences in various individual sports. For example, singles tennis and badminton are "one-on-one" individual sports, while swimming and running involve many individuals racing against each other for a best time. And, while tennis and badminton have international "bracketed" tournaments, artistic gymnastics has competitions where all athletes perform and are scored by judges individually, with the highest score winning at the end of the competition. Another individual sport, Golf, involves many players competing for the lowest score on a particular course over several days. Teams had to address these differences and how their model could be adjusted to account for these variations.

Lastly, the Challenge required a discussion of any changes to a team's G.O.A.T model for individual sports to extend it to a G.O.A.T model for a team sport. Teams were not required to develop the model, but rather to discuss required revisions. Better teams identified the difficulty in recognizing an individual as a G.O.A.T. when teams win or lose based on the entire team. Does the team with one or more superstars always win? Are those superstars always G.O.A.T. candidates? Can a particular sport's G.O.A.T. be on a losing team? Factors for the G.O.A.T. in a team sport might include measuring the impact of that player on their team or on the sport itself, or comparing that player to other players of that same position. Better teams provided a general description of the revision process with examples of some of the different factors and considerations required.

A Note on Sensitivity Analysis. Sensitivity analysis is always important, but given the context of this Challenge, sensitivity analysis was critical: How sensitive is your identification of the G.O.A.T. to a small change in one of the variables or coefficients or weightings you chose? The results of this analysis were an important part of a good report. This analysis section should also include an error analysis and a discussion of strengths and limitations of the model.

Letter. For 2021, teams wrote a letter to the Director of Top Sport describing their model for finding the G.O.A.T. of the selected team sport. As the Director is likely an executive who may not understand a lot about mathematical modeling or science, the letter needed to present general principles and key findings in an understandable way.

General Structure and Presentation. Overall, submissions should be clear, organized, and well presented. Tables, graphs, and flow-charts sometimes help to illustrate a complicated model or process. Teams should document and identify any resources used. Ensure that your paper concludes with a short summary of the actual solution or findings to the requirements of the problem. The solution report should be at most 20 pages long, plus a one-page summary sheet, table of contents, and one-page letter. Since the evaluation process occurs under a blind review protocol, teams should not include identification notes (names, schools, or country/ region).

Results and Recommendations

Of the 51 papers judged in the International Round, 17 were judged Successful, 26 Honorable Mention, 6 Meritorious, and 2 Outstanding. But, more importantly, significantly more than 51 teams worldwide were brave enough to take this year's Challenge—which for all these teams is praiseworthy! By reading the reports, especially the two Outstanding ones, the philosophy of IM^{2}C is clear: different approaches, inventive and creative ideas, application of the mathematics students know, and the use of various relevant aspects can all lead to great results in modeling.

In general, most teams made good choices for the factors considered crucial in their models. The number of matches and tournaments won could of course not be overlooked, but also some other relevant factors in deciding the G.O.A.T were considered. One of the teams tried to measure popularity by counting Google-hits; quite an original idea! Finding a suitable way to weigh all factors was crucial for a good model. The choice of factors and the use of particular information (facts, figures, and graphs found on the Internet) helped teams to develop their model and present it clearly.

Justifying the information teams chose to use was very important for the IM^{2}C judges. Given that ranking and rating methods exist for a variety of sports, judges looked for teams to create their own models. If a team used an existing system, they had to examine each component of the model and assess whether it was consistent with the objectives of their G.O.A.T. model, or they had to significantly adjust the existing system's model in applying it to the Challenge problem.

As in previous years, some teams included extensive computer codes. While the use of computer code can be very helpful, IM^{2}C does not require actually including the computer code in the report. A good description of the working of the code, and the choices made in constructing it (as being a vital part of the modeling process) should be part of the report. Teams may include the code itself in an appendix (judges will not read the code in detail). The most important thing in the report must be the model itself and the underlying choices made in developing it. And finally, teams should realize modeling is not about trying to use the most complicated mathematics, rather it is about describing a real-life situation by use of appropriate mathematics that team members understand and can explain. We are happy to see reports continue to improve as most teams prepare themselves by looking at previous IM^{2}C papers.

General Advice to Teams Participating in Future IM^{2}C

The IM^{2}C is definitely a challenge. Teams have to organize themselves, address all requirements of the problem, and write a report in a short period of time. Budgeting time becomes critical so that you leave enough time to effectively communicate your work and results to the Challenge judges.

Our advice is to allow plenty of time to construct your report. In fact, consider outlining the report as soon as you begin working on the problem. This outline will guide your team in its work and provide a logical path to your solution for readers of your report. Remember, you are communicating with judges from many countries of the world. The judges are not necessarily familiar with the curricula of your school, so present the development of your model in a logical and easily understood fashion. Judges are not looking for the papers that use the most sophisticated mathematics. Do not force the mathematics upon a given scenario. Rather, begin with the simplest mathematics that solves the problem you have identified and use mathematics that you understand. Later, as appropriate, refine and enhance your model to increase its precision, or adjust your assumptions to find a more broadly appropriate solution.

Pictures, graphs, tables, and schedules can be quite effective and efficient in communicating your ideas. The use of relevant pictures and graphs can make a report clearer and more pleasant to read. Your report should include a combination of various representations: symbolic, graphical, and text that best present your model and solution. Realize, however, that large tables and extensive code or data might be better as supporting material located in an appendix.

The use of symbolic formulae and algorithms are quite essential in a mathematical modeling assignment. The use of unexplained formulae, however, will not make the report more convincing. The reader needs to believe that the writers themselves understand the formulae used. This is done through explanations and analyses of your modeling processes. The readers of your report, while experienced mathematicians, are not experts in all parts of the great world of mathematics.

Appendices can be very useful, but do not expect the judges to read them. While judges may refer to an appendix to check a reference or to get a general idea of your computer code, they will not fully read the appendices. Therefore, do not place anything critically important to the development of your model in an appendix.

Remember to list any sources you used during your work on the Challenge and to document in your paper where you used these sources (e.g. a graph or picture from a particular web site). Follow the rules for completing your solution report within the specified number of pages and in a font size of no smaller than 12-point type.

Overall, present the development and analysis of your model in a manner that a wide audience could understand. Consider who might be using your model and explain your model to that audience, as well as the judges. Ensure you close your report with a conclusion and a summary of your results.

Finally, the IM^{2}C judges extend our highest praises and compliment all teams on their efforts. In this year when the pandemic significantly impacted all school life, it is impressive to see so many students from all over the world find time to join this year's Challenge. We thank all schools, teachers and advisors for making it possible for students to participate in IM^{2}C 2021. Teams did a great job in developing a system to decide on the G.O.A.T in all kinds of sports. The judges (all mathematicians and teachers) saw many creative approaches and had several stimulating discussions about the papers submitted. It is a pleasure to see that many students across the globe are involved in mathematical modeling—and they seem to like it! Well Done!

For more information about the IM^{2}C, including the complete 2015–2021 results and sample papers, visit

www.immchallenge.org

Who Is The Greatest?

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Advisor:

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TAN YI KAI

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TAN JIECONG

Summary Sheet

Who is the Greatest of All Time (G.O.A.T.)? This has always been a point of contention in the world of sports among athletes, sporting organisations, and fans. This is because different entities have different ways of judging sports figures, often due to inherent bias based on personal preference or nationality, which causes them to gravitate towards their favourite athletes. We may also be affected by recency bias, where we hold newer athletes in higher regard as we can remember their feats better.

Our model introduces an objective and reliable approach to determine which athlete is the greatest based on a single year's performance, as well as the G.O.A.T. of individual or team sports, spanning decades of sporting history.

For Task 1, we made use of the results of the four Grand Slam tournaments in 2018, to determine the greatest female tennis player of 2018. We developed a model based on a weighted directed graph to represent the different matches played by different players, who act as nodes in the graph. In order to find the relative ability of players, we used the Floyd-Warshall Shortest Path algorithm to predict the results of matches between players who did not play against each other. Hence, we can obtain the weighted out-degree to in-degree ratio for an accurate measure of the ability of a player, based on the predicted win-to-loss ratio. Hence, we accurately determined the greatest female tennis player, which agrees with the Women's Tennis Association (WTA)'s 2018 ranking.

For Task 2, we extended our model in Task 1 to include other factors which have to be considered in order to determine the G.O.A.T. of an individual sport. For Task 2a, we chose men's 200-metre butterfly (swimming) to use as an example. As our model uses a weighted directed graph, it can objectively measure the ability of athletes, by comparing the difference in timings within every race. In addition to the relative ability of players, we also incorporated a much wider range of factors, such as the athlete's medal tally, consistency of performance and strength of rivalry with other athletes.

For both Task 1 and 2a, we successfully obtained the greatest player, based on the degree ratio and the G.O.A.T.-ness score defined. The greatest players were easily distinguishable from the rest of the athletes due to their significantly higher scores, which shows that our model is indeed effective in sieving out the best athletes. This also demonstrates that our model is applicable to both one-on-one sports (tennis) and sports with inanimate standards (swimming), and can be easily adapted for any individual sport (Task 2b) by making slight adjustments to the weightage of the different factors used in Task 2a.

Our model in Task 2a can also be modified to find the G.O.A.T. of a team sport (Task 3), by measuring the relative ability of teams, to find the strength of each team. This is to be complemented by the individual contribution of each player. Thus, we can effectively select the best players from the best teams to determine the G.O.A.T. for a team sport.

Contents

I. Letter to Director

Dear Director,

Thank you for your confidence in our team. After conducting extensive testing and research, we would now like to present our model to you. Having tested our model on the men’s 200-metre butterfly to ensure the model's accuracy and reliability, we will demonstrate how our model has successfully determined the G.O.A.T. for men’s 200-metre butterfly.

Our model considers a few key factors that are highly useful for determining the G.O.A.T. of a certain category of individual sports. These are: the athlete's relative ability to other competitors, the athlete's consistency, his or her medal tally, the number of times he or she has broken the world record, and finally, the intensity of rivalry with other competitors. An athlete's relative ability was obtained by modelling the matches between players as interactions with differing strengths.

All of these factors are given a different weightage in the final equation to calculate the “G.O.A.T-ness” of each player, based on their relative importance compared to other factors. For instance, for 200-metre butterfly, we decided that medal count would be given a higher weightage as compared to strength of rivalry, since the athlete's number of medals directly reflects his or her exceptional abilities in the field.

Our model has determined Michael Phelps to be the G.O.A.T. for this event, with his "G.O.A.T.-ness" score being significantly larger than the next best swimmer. This is highly reasonable given his outstanding performance, impressive medal tally, and the fact that many news organisations recognised him as the G.O.A.T. As such, our model is able to objectively and conclusively identify the G.O.A.T. of various sports.

Furthermore, our model not only considered factors that are featured in various world athlete rankings, but also rivalry and consistency, which carry great importance when determining the G.O.A.T. This ensures that the G.O.A.T. identified is not a newcomer or a rising star, but instead one who has been consistently outperforming other athletes in the sport.

We hope that you have gained a better understanding of how our model works, and that you will strongly consider adopting our model to determine the G.O.A.T. for various categories of sports in the future. If you have any further queries about our model, please feel free to contact us, and we will be very pleased to share more details with you.

Regards,
Team XX

1. Introduction

The term "G.O.A.T." - the *Greatest of All Time* - has been popularised in recent years, with ardent sports fans constantly debating which professional athlete deserves such a prestigious title. Yet, there is a wide range of factors that make it difficult for sports commentators to reach a consensus on who the G.O.A.T. is, including the athletes' positions on world sports rankings, number of matches won, and other noteworthy achievements such as world records. Thus, in this study, we aim to develop a model that determines the G.O.A.T. for different individual and team sports as objectively as possible, based on competition performance and other unique achievements.

Current athlete rankings for various sports, ranging from individual games such as chess to team sports such as soccer, include a set of common factors. For instance, world rankings by World Athletics are based on the measured results of athletes and their placing during competitions, with extra points given to athletes who achieve new world records [1].

Also, the Fédération Internationale de Football Association (FIFA) uses a ranking system where each team has its own rating point. Points are evaluated based on several factors, including the strength of opponent (which is based on the opponent's ranking), the match importance (based on the type of competition), and the outcome. There is also different weightage given for scores attained at different timings; the weightage of more recent scores is higher [2].

Our work thus takes into account the more relevant factors mentioned above, and splits them into "objective performance factors" and "subjective perception factors". Thus, our model can be applied more realistically to various sports, which may depend heavily on one type of factor more than another.

Therefore, to determine the G.O.A.T. of various sports, we developed an algorithm that is easily adaptable to suit the nature of results for many sports. In addition, we have conducted test cases to assess the accuracy, viability and sensitivity of our model.

2. Restatement of Problem

Top Sport, a sports network, has requested our team to develop a model for individual sports, and use it to determine the G.O.A.T. of one individual sport of our choice.

This problem requires us to complete 4 tasks:

1. Determine the greatest woman tennis player of 2018 based on the results of the four Grand Slam tournaments in 2018, using a model.
2. Determine the G.O.A.T. of any individual sport of our choice using a model, and discuss the changes required for this model to determine the G.O.A.T. of other individual sports.
3. Discuss the changes required for the model from (2) to determine the G.O.A.T. of team sports.

4. Write a letter to the Director of Top Sport to describe our model and the G.O.A.T determined in (2).

3. Task 1

3.1. Assumptions

In order to simplify the model and make it possible to be computed mathematically, assumptions would have to be made. These are the key assumptions that we made, and their justifications:

1. Walkover matches are not factored into the model.
   Justification: Walkover matches are not an indication of one's ability of reputation, as they are mostly a result of injuries and other similar reasons against the player's will [3].
2. The tennis player's match performance is an accurate representation of the player's tennis abilities. The players' are equally well-rested before each match.
   Justification: The most significant determinant of a player's performance in each match is likely to be his or her skills/abilities, and other conditions of the athletes, such as fatigue and emotional stress, are difficult to account for because such factors cannot be easily quantified.
3. The home-field advantage, which refers to the tendency for sports performers to win more often when competing at their home facility, of the female tennis players is low and negligible.
   Justification: A research study found that although some degree of home advantage exists for men's tennis, the performance of female tennis players appears to be unaffected by home advantage [4]. In addition, the impact of home advantage is difficult to quantify, as the extent of home advantage can vary across different athletes. As we intended our model to be a more objective form of assessment, we excluded this factor from our model.

3.2. Variables

To compare the abilities of the female tennis players in the four Grand Slam tournaments in 2018, we create a function d_u to evaluate the performance of these players within the scope of the four tournaments given. This function would take into account the following variables:

Variable`	Definition
u_i, v_i	The two players involved in match i
$m_{u,v}$	Total number of matches played between player u and player v
s_{u_i}	Score of player u for match i
$g(u, v)$	Difference in the total score of player u and player v for match i i.e. $\sum s_{u_i} - \sum s_{v_i}$; if $g(u_i, v_i)$ is positive, this represents a directed edge from u to v and vice versa
$h(u, v)$	Average winning margin between player u and player v based on number of matches

Variable`	Definition
d_u	Degree ratio $\frac{\sum_{i=1}^{n} h(u,i)}{\sum_{j=1}^{m} h(j,u)}$ i.e. the ratio of the sum of all the weights of outward edges to the sum of all the weights of inward edges

3.3. Model Development

Factors affecting the greatness of a tennis player

These are the factors we have considered to determine the greatness of a tennis player:

- **The margin that a player wins another player by**
  By winning another player by a larger margin, it indicates that the winner has a much greater skill level relative to her opponent, as the winner can defeat her opponent with greater ease.
- **The number of players that a player wins**
  If a player can win more opponents, her ability should be ranked higher than others as she is more capable. In addition, winning more players indicates that she is likely to have progressed to later stages in the tournaments, such as semi-finals or finals, and this is an indicator of greatness.

We did not make use of the seed rankings provided, as that would defeat the purposes of our model, since it would already introduce some bias in our model beforehand. This is to ensure greater objectivity and that our rankings are solely based on the results of the 2018 tournaments.

Assessing skill of player using weighted directed graph

When determining the greatest female tennis player of 2018, it is critical to assess the performance of the athlete with respect to other athletes competing in the same sport. Hence, we modelled the matches played between different players as a directed graph of nodes and directed edges. Each node represents a player, and each directed edge is drawn from the more proficient player to the less proficient player. This is determined by the polarity of $g(u_i, v_i)$, which is given by the following equation:

$$g(u,v) = \sum s_{u_i} - \sum s_{v_i}$$

$g(u,v)$ represents the **total winning margin between player u and player v**, which can be represented by the difference between $\sum s_{u_i}$, the total sum of player u's scores in the matches played between player u and v, and $\sum s_{v_i}$, the total sum of scores of player v in these same matches. A positive value will indicate that player u has a positive net margin over player v, which indicates that player u has a higher skill level than player v.

To compare this relative skill level with other players, from $g(u,v)$, we averaged out the margins based on $m_{u,v}$, the **number of matches played between player u and v**, as shown by the following equation:

$$h(u,v) = \frac{g(u,v)}{m_{u,v}}$$

$h(u,v)$ will form the edge weight in our directed network. All edges will have positive edge weight, and hence the direction will represent the polarity of $h(u,v)$. In other words, if $h(u,v)$ is positive, the edge will be directed from player u to player v. If $h(u,v)$ is negative, the edge will be directed from player v to player u.

By averaging $g(u,v)$, we ensure that the number of matches played has little effect on our edge weight, as this will be purely a gauge of relative ability of one player compared to another, in order to determine who is better, and how much better. Hence if a player plays many matches with another player, and wins many of them, she will not be disproportionately rewarded in our model, especially if the other player is not as skilful and they play many matches with each other.

Filling in missing information using the Floyd-Warshall Algorithm

As players do not play with all other players in the tournament, it can be difficult to determine the relative skill level of players if they do not play a match. In order to fill in the missing information, we used the Floyd-Warshall algorithm, which is an All-Pairs Shortest Path (APSP) algorithm that finds the shortest path in a directed weighted graph between all possible pairs of nodes (i.e. players) in the graph.

The Floyd-Warshall algorithm can be used to add edges (or relationships) between players even if players have not played with each other before. It assumes the worst-case scenario, to determine the minimum margin player u wins player v by. If this minimum margin is higher than the minimum margin player v wins player u by, then player u is more likely to win player v, and vice versa. This can be determined using the Floyd-Warshall shortest path algorithm on the directed graph to determine the shortest path between two players u and v, by taking edge distance to be the weight of the directed edge, given by $h(u,v)$. If no directed edge exists from player u to player v, the edge weight is set to zero so that player u never wins player v in any scenario.

Using the new edge weights obtained from the Floyd-Warshall algorithm, we reconstruct all the possible edges between all players, in a new directed graph. In this new directed graph, the edges are directed from the winning player to the losing player, which includes our predictions for players who have never played with each other before. This allows us to predict the margins between such pairs of players in the 4 Grand Slam tournaments, hence compensating for the lack of data and greatly increasing our accuracy in finding the player who is most consistently outperforming others, as we can determine the proportion of players a particular athlete can win more accurately.

For each player u, we can then calculate the **degree ratio** d_u, which is the weighted ratio of out-edges against in-edges of the directed graph. This is an indicator of the probability of winning, which accounts for both the frequency that the player wins at, as well as the margin that the player wins by, which are both essential factors when gauging the ability of an athlete. The degree ratio is given by the following formula:

$$d_u = \frac{1 + \sum_{i=1}^{n} h(u,i)}{1 + \sum_{j=1}^{m} h(j,u)}$$

A constant of 1 was added to the numerator and denominator so that d_u is always defined.

The weighted degree ratio is an accurate indicator of ability, because it represents the predicted win-to-loss ratio of each player, which is a gauge of a player's ability to defeat other players in the sport.

3.4. Results

We ranked the tennis players based on their calculated degree ratio, d_u. A higher degree ratio is reflective of a more skilled tennis player, because she has a higher total probability of winning against other players, and a lower total probability of losing against other players.

Actual WTA Ranking (2018)	Ranking based on our model	Name of tennis player	Degree ratio
1	1	Simona Halep	204.83
2	2	Angelique Kerber	83.67
5	3	Naomi Osaka	62.07

Table 1: Our ranking of the top 3 female tennis players based on degree ratio

As shown in Table 1, we found that the greatest female tennis player of 2018 was Simona Halep. Also, our ranking of the top 3 female tennis players in 2018 closely matches that of the actual Women's Tennis Association (WTA) rankings [5]. This shows that the accuracy of our model is high, and is an objective method of analysing the players' abilities solely based on 2018 results, instead of judging the players based on past performance. In fact, Naomi Osaka, who won her first ever Grand Slam title in the 2018 US Open [6], was featured highly in our rankings, showing that our results are indeed determined on the basis of 2018 results.

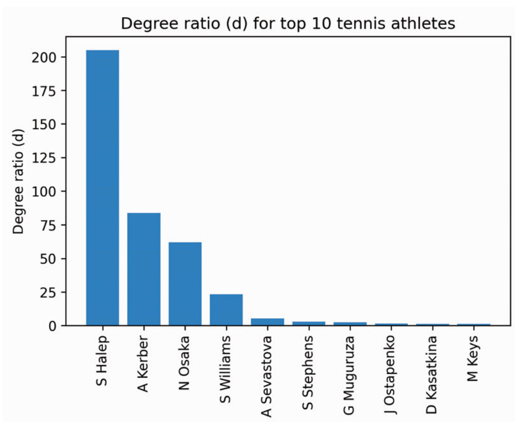

Fig. 1: Degree ratio for top 10 athletes in women's singles tennis in 2018

As shown in Fig. 1 above, our results are deterministic as there is an extremely large difference in the value of d_i between Simona Halep, and the rest of the players. We can thus conclusively determine, without much doubt, that she is the greatest player for 2018 women's singles tennis.

However, some discrepancies still arise because the WTA takes into account the points earned at every tournament during a 52-week stretch, while our model only takes into account the results of the four Grand Slam tournaments.

Also, the number of points awarded for each tournament are determined by how far players advance, and thus accounts for preliminary rounds, such as the round-of-128 and round-of-64, while our model only looks at the results for the round-of-16 and onwards.

Fig. 2 below shows the graph obtained from the raw data of the matches between different players. The red node represents Simona Halep, the best female tennis player in 2018 according to both our model's rankings and WTA's 2018 rankings. The green nodes represent the players that Simona Halep won in the four Grand Slam tournaments, such as Sloane Stephens and Angelique Kerber. The directed edge is drawn from the winning player to the losing player, and the thickness of the edge is an indicator of edge weight. The network representation of our model is consistent with our results, which suggests that Simona Halep is consistently outperforming many of the most skilled players in singles women's tennis by significant margins.

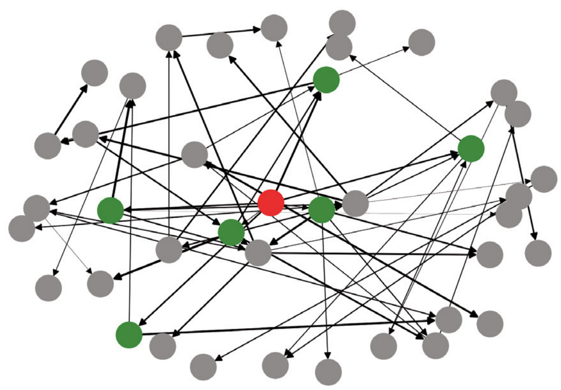

Fig. 2: Weighted directed graph of athletes competing in women's singles tennis in 2018 before running the Floyd-Warshall algorithm

3.5. Strengths of Model

Our network representation is advantageous as it can **represent many more relationships between different players** as compared to a simple hierarchical scoring method. By averaging our edge weights over the number of matches, we can obtain a fairly accurate understanding of the relative abilities between players.

Via the Floyd-Warshall algorithm, our model also extrapolates the data to estimate the probability of each tennis player winning against all other tennis players. This is because each player does not get the chance to play against every other player, thus our model offers a more definitive ranking of the players, as compared to only using the match results between players who have competed against each other before. If we had not filled up the missing edges, we may be imposing an unfair penalty on athletes who lost in the semifinals and finals. These athletes are skilled, but they lost to other more skilled competitors. Hence, the Floyd-Warshall algorithm is essential to **ensure fairness** in our evaluation of the relative ability of athletes.

Compared to heuristic models, which includes more randomness, our model is **deterministic**, which means that it always executes in a similar fashion and produces the same answer. This greatly increases the credibility and validity of our model, such that *Top Sport* will be more inclined to adopt it.

Our results are able to conclusively determine the greatest player, in this case Simona Halep, with a clear distinction from the rest of the players using our method. This indicates that Simona Halep is undoubtedly the greatest female single's tennis player of 2018.

3.6. Limitations of Model

However, it should be noted that the Floyd-Warshall algorithm is **computationally intensive**, as it has a time complexity of $O(N^3)$. where N is the total number of tennis players in the data set. Hence, while this algorithm may have been feasible for this data set, for sports with a much larger number of players (e.g. more than 1000 distinct players), this limitation may be more pronounced.

4. Task 2

4.1. Chosen Sport (Task 2a)

For this task, the sport that we chose is the men's 200-metre butterfly event in swimming. This choice was made due to the differences to the mode of competition for tennis. This is because tennis is a "one-on-one" match-based competition between two players, allowing edges to be easily drawn between players based on match results, whereas swimmers compete based on an inanimate standard (the fastest swimmer wins), adding an additional layer of difficulty and complexity for our model, and demonstrating the vast applicability of our algorithm.

We have selected two prestigious international competitions to obtain data from, the Olympic Games [7] and FINA World Championships [8], over a time period of 1990 to 2020, so as to compare a large number of athletes from all nationalities and across several decades. We chose these two competitions as they have the most complete sets of data over the specified time period, and include the results from both finals and semifinals, allowing us to make use of a larger range of data.

4.2. Additional Assumptions

On top of the assumptions made in Section 3.1 that the matches are accurate representations of ability, and there is negligible home field advantage, we need to make further assumptions:

- Swimmers who have been embroiled in severe scandals (e.g. use of performance-enhancing drugs) are not worthy of the G.O.A.T. title due to poor morals, and are thus excluded from consideration in our model.
  Justification: Athletes involved in scandals often gain large negative press and have a poor reputation [9] as they are viewed to have violated the principles of sportsmanship and integrity, thus the public will very likely not support these athletes. Furthermore, many athletes have been banned from competing when caught using drugs, as doping is prohibited by most international sports organisations, including the International Olympic Committee [10].

- We also assumed that the G.O.A.T. will be found within our dataset.
  Justification: The Olympic Games and FINA World Championships are two of the most prestigious international sporting events in the world, hence only the best swimmers from around the world will be able to stand a chance to compete in these two competitions, since countries will only send their best athletes to compete.

- The sports technology used by athletes competing against each other has negligible impact on the relative performance of athletes in the same match. An example of technology used in sports include better swimwear to reduce water resistance. The component of our model to assess athlete ability only draws relations between athletes who have competed against each other, hence we accounted for the improvements in sports technology over time.
  Justification: While it is true that sports technology can give certain athletes an edge over others, and does not accurately represent the ability of athletes, this effect is negligible as countries that athletes are representing often invest in their athletes and provide them with the best possible technology so that they stand the best chance of winning. Therefore the effect on differing levels of equipment and technology is negligible.

4.3. Additional Factors

Given the need to consider a much larger database of results from a variety of competitions, we will introduce additional variables, which can be considered "subjective perception factors", to build on to our previous model, as shown below:

- **Prestige of competition -** Usually, sporting competitions are categorised according to the level and significance of the competition, and the scores achieved in different competitions are given different weightage [11]. The highest category reflects the strongest competitions and consequently awards the most points. For example, the weightage given to the Olympic Games are higher than those at other local competitions.
  According to the categories determined by World Athletics [12], the Olympic Games and various World Championships are in the same category, which shows that they are likely to be equally prestigious. Thus, in our model, we will be according the results attained at the Olympic games and FINA World Championships the same weightage.

- **Special achievements such as World Records -** Athletes who attain a new world record are likely to receive greater publicity than just winning first place, allowing them to gain more recognition in the sport scene. Also, witnessing world records being broken brings great pleasure for athletics fans, thus these athletes would leave a greater impression in the public's eyes. Therefore, such once-in-a-lifetime achievements should be taken into account when determining the G.O.A.T. of various sports, and various sports world rankings such as World Athletics do so too - "bonus points are given as an extra reward for the obvious significance and promotional value of such performance" [13].

- **Medal tally -** The number of medals an athlete receives over time, the more public recognition the athlete receives, as winning first, second or third place across multiple years is a clear signal of these athletes' superior skills compared to other athletes. Thus, the accumulated medal tally should contribute to how likely the public would perceive the athlete as a G.O.A.T. Furthermore, prize-giving ceremonies for various sporting competitions are often heavily publicised, thus helping the athletes to gain fame and respect as well.

- **Consistency of performance -** Athletes who are able to maintain or even improve their skills over time would receive more media hype and public attention for various competitions, as they are consistently viewed as the most likely to win. Furthermore, being able to sustain their performance is testament to their commitment and dedication to the sport, as well as their perseverance and undying spirit, which are hallmarks of a G.O.A.T. Hence, tracking the consistency of athletes' performance across multiple competitions is crucial.

- **Famous rivalry -** Famous rivalries are critical when it comes to swimming, and make swimming matches and the athletes memorable. Rivalries often gain a lot of publicity, and are heavily advertised in the media. Hence, when athletes compete with their rivals, and win the race (often by a fraction of a second), they are considered to be the greatest, especially in comparison to rivals who are already one of the best in the sport. Furthermore, according to the social identity theory, sports fans seek membership in groups that will positively reflect on their self and public image [14], thus public support for both athletes in a strong rivalry will increase greatly, as compared to other athletes.

Variable / Function	Definition
G_i	Measure of "G.O.A.T.-ness"
d_i	Measure of athlete's ability, i.e. weighted degree ratio
b_i	Weighted medal tally based on gold, silver and bronze medals
r_i	Number of times the world record was broken by the athlete
c_i	Measure of consistency

Variable / Function	Definition
$\beta_{i,j}$	Average margin of victory between an unordered pair of 2 athletes
α_i	Rivalry score
$norm(x_i, y, z)$	Normalisation function for variable x_i in range $[y, z]$ This is given by $y + ((z - y) * \frac{x_i - min(x_i)}{max(x_i) - min(x_i)})$

4.4. Model Development

Our overall equation for calculating the "G.O.A.T.-ness" of an athlete is as follows:

$$G_i = d_i \times norm(c_i, 1, 1.5) \times norm(\sqrt{b_i}, 1, 2) \times norm(\alpha_i, 1, 1.25) \times (\sqrt{r_i} + 1)$$

Measuring the ability of an athlete

The ability of an athlete is part of what makes an athlete great. In order to stand out from others, an athlete not only has to do well individually, but has to outperform others to be labelled as great. Hence, our model takes into account one's relative performance to others.

d_i represents the **weighted degree ratio**, which corresponds to the **ability score** of each swimmer. This was calculated using a similar method as Section 3.3 (Task 1). For each race every year, we used the directed graph as explained in Section 3.3. We used data from the semi-finals and finals of each competition, and each race has 8 athletes. For earlier years, the finals had 16 athletes, so we only included the top 8 for each round for fairness. For each race, based on the timings, we constructed a directed edge from athlete u to athlete v if athlete u had a better timing than athlete v during the race. The edge weight is defined as the average of all the margins. In this context, margins refer to the difference in timings for the men's 200-metre butterfly event. This method has been elaborated on in Section 3.3.

Using the model in Section 3.3, we can calculate the weighted degree ratio d_i. Note that the Floyd-Warshall algorithm was not used, as the graph generated was much denser as compared to the graph for women's tennis in Task 1, since swimming is a sport with an inanimate standard and not "one-on-one" sport like tennis. A dense graph refers to a graph where number of edges in a graph is close to the maximum number of edges in a fully connected graph [15]. Hence, there was no need to fill up missing relationships in the graph.

In addition, unlike Task 1, where all the matches occurred in 2018, the competitions in Task 2a spanned over a longer period of time, and it would be unfair to directly compare different athletes from different eras based on their raw timings, without accounting for general improvements over the years. Hence, by not using the Floyd-Warshall algorithm, our model only compares each athlete with other athletes in the same era for fairness of comparison.

Calculating weighted medal tally

Medals are an important form of recognition for an athlete's achievement. Thus, for **medal tally** b_i, we will calculate the total number of "medal points" each swimmer had earned from both the FINA World Championships and the Olympic Games, from 1990 to 2020. Using the existing "sum-ranking system" for a weighted medal tally [16], a gold medal is worth 3 "points", a silver is worth 2 "points", and a bronze is worth 1 "point". This value is then square-rooted to reflect the diminishing returns from each additional "medal point".

Then, for these values to be multiplied to the overall function to obtain a measure of "G.O.A.T.-ness", they will be normalised to a scale of 1 to 2. Thus, an athlete's normalised medal tally represents how close or far he is from the best and the worst athlete, in terms of medal tally. A normalised value of 1 represents the worst medal tally in the data set, and will not affect the value of G_i when multiplied to it, while 2 represents the best medal tally, and will rather significantly amplify the value of G_i.

Number of world records broken

The number of world records broken is an important factor, as it usually makes the news, allowing the athlete to gain international attention. Thus, it is often an athlete's pathway to fame. For the number of world records broken, we chose to represent this variable as $\sqrt{r_i} + 1$, where r_i is the **number of times the world record for men's 200-metre butterfly was broken** by the athlete.

If $r_i = 0$, $\sqrt{r_i} + 1 = 1$, thus the value of G_i will not be affected. However, if $r_i \geqslant 1$, the value of G_i will increase at a decreasing rate, to reflect the decreasing significance of each additional time an athlete breaks the world record. When the athlete breaks the world record the first time, he or she is already regarded as having exceptional skills, and subsequent world records will add to this public image, but with less significance, due to the pre-conceived expectation that the athlete is already extremely talented.

Measurement of consistency

Consistency is critical to ensure that the athlete is truly the Greatest of *All Time*, and not just a one-time phenomenon. The **measure of consistency**, c_i, is based on the number of competitions, or specifically finals, that the athlete had participated in. This indicates that the athlete has been participating in the sporting event for a long period of time, and has sufficient experience in competitions to consistently enter the finals, so that he or she can be highly regarded as one of the greatest athletes of "*all time*", and not just the greatest athlete for a short period of time.

This value, c_i, is then normalised between 1 to 1.5. The scale of normalisation is not as large as that for medal tally b_i due to the interdependency of our chosen factors, as entering the finals more times would naturally translate into a higher chance of attaining more medals. Thus, a small aspect of consistency would have already been accounted for in b_i, the normalised weighted medal count.

Measure of strength of rivalry

$\beta_{i,j}$ between two athletes is defined as the **average margin** (i.e. difference in timing) between them when they are in the same race. A narrow win would be much more significant in the eyes of the public, since it implies that both athletes are similar in ability and there is a possibility the athlete who lost would be able to win in the future. Hence, the smaller the margin, the higher the rivalry score. We considered the average victory margin between medallists. For each athlete, we identified his main rival by finding the opponent for which he has the smallest winning margin. For athlete i with opponents of index 1 to n, the rivalry score of athlete i, α_i, is given by:

$$\alpha_i = max(0, 1-2\beta_{i,1}, 1-2\beta_{i,2}, \ldots, 1-2\beta_{i,n})$$

$$= max(0, 1-2\beta_{i,j}) \ \ \forall j \in [1,n], j \in \mathbb{Z}^+$$

Therefore, the rivalry score α_i is given by the strength of rivalry with athlete i's strongest rival. We acknowledge that this is a minor factor as compared to other factors we have previously highlighted in this study, as the athletes' performance is more important, hence a lower normalised weight is placed on the rivalry score (from 1 to 1.25).

The strength of rivalry between 2 athletes decreases as the average win margin increases. We proposed a linear relationship between the rivalry score α_i of an athlete, and $\beta_{i,j}$, the minimum margin the athlete defeats his strongest rival by. When the margin is equal to or above 0.5s, the score is set to 0, as it will be a clear win in that case without any significant rivalry. Since the maximum of the rivalry score α_i is 1, and α_i is 0 when $\beta_{i,j}$ is 0.5, we can represent this relationship using the expression $1-2\beta_{i,j}$. The maximum rivalry, α_i, is the maximum of all $1-2\beta_{i,j}$.

4.5. Results

Firstly, to analyse our values obtained for the weighted, normalised medal tally, these are the different number of medals, and values for b_i and $norm(\sqrt{b_i}, 1, 2)$ that we obtained for the more notable swimmers, from the period of 1990 to 2020 and in the 2 competitions for men's 200-metre butterfly:

	No. of gold medals	No. of silver medals	No. of bronze medals	b_i	$\sqrt{b_i}$	$norm(\sqrt{b_i}, 1, 2)$
Michael Phelps	8	1	0	26	5.10	2.0
Melvin Stewart	2	0	0	6	2.45	1.48
Kristóf Milák	1	0	0	3	1.73	1.34

Table 2: Raw data and derived data for top 3 athletes

As shown in the table above, as Michael Phelps has the highest medal tally among all the swimmers, his value of $norm(\sqrt{b_i},1,2)$ is the highest, at 2.0. Those without any medals would have a value of 1.0.

Using our model, these are the values of the different variables that we have obtained, for the athletes with highest G.O.A.T.-ness scores, G_i. Among the top 10 athletes we identified, 4 of them (Michael Phelps, Melvin Stewart, Chad Le Clos, Michael Gross) were ranked among top 6 medallists in the Olympic Games' website [7], thus showing our model's accuracy:

	d_i	$norm(c_i,1,1.5)$	$norm(\sqrt{b_i},1,2)$	$\sqrt{r_i}+1$	$norm(\alpha_i,1,1.25)$	G_i
Michael Phelps	77.30	1.5	2.0	3	1.23	855.76
Melvin Stewart	43.45	1.1	1.48	2	1.0	141.51
Kristóf Milák	31.83	1.05	1.34	2	1.0	89.54

Table 3: Variables for computation of G_i

As shown above in Table 3, our model has determined that Michael Phelps is the G.O.A.T. of men's 200-metre butterfly, as he has the highest value for G_i, and this is not surprising given his exceptional performance in most of the variables that we have selected. As he had taken part in 10 finals from 1990 to 2020, the most for any swimmer in this event, his normalised measure of consistency is the highest. Furthermore, having broken the world record for the men's 200-metre butterfly for a total of 4 times in these competitions ($r_i = 4$), his value of $\sqrt{r_i}+1$ is also the highest.

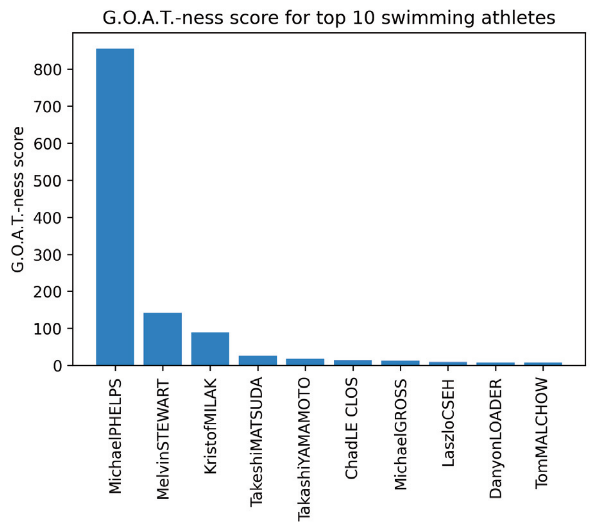

Fig. 3: G.O.A.T.-ness score for top 10 male swimmers in 200-metre butterfly

As shown in Fig. 3, we also observe that the results of our model show similar patterns as Task 1, where the G.O.A.T. has significantly greater ability score d_i, and also significantly greater G_i than the other athletes, which indicates that our model is deterministic as elaborated on earlier. Hence, our model allows us to identify the G.O.A.T. with high certainty. The network of swimmers can be found in the Appendix.

Indeed, several news organisations, including FOX Sports, USA Today Sports, and The New York Times [17, 18, 19], have recognised Michael Phelps as the G.O.A.T. of swimming, or one of the best athletes of all time, due to his "longevity" in the field, and his high medal count and number of world records.

4.6. Strengths and Limitations of Model

Strengths

Our model takes into account several quantifiable factors and splits them into 2 categories - firstly, the differences between the swimmers' timings which allow us to have an **objective analysis of their relative abilities**, and secondly, factors that would shape one's perception of the swimmers, such as their consistency and number of world records. This is crucial for us to gain a **comprehensive overview** of the swimmers' achievements and abilities over a long period of time. Besides including factors that are featured in various global rankings of athletes, which demonstrates our model's relevance to real life, our model also considers time-based factors, such as consistency, which carry great importance when determining the G.O.A.T. This ensures that the G.O.A.T. identified is not a newcomer or a rising star, but instead, one that has been consistently outperforming other athletes in the sport.

Our model also takes into account that the ability of players improve over time [20]. This can be due to advancements in sports science such as better swimwear, more calibrated diets of athletes, and also specially tailored training regimes, leading to a general improvement in the quality and abilities of athletes. In order to factor this into our model, we ensured that edges were only drawn between athletes in the same competition, so that the abilities are compared on a relative basis instead of using an absolute scale.

Also, our model is **flexible** and can be easily adapted for the analysis of different types of sports, which includes both one-on-one sports like tennis, as well as sports with an inanimate standard such as swimming, by adjusting our directed graph model.

Furthermore, our model made use of **real-life data** from the Olympics Games and FIFA World Championships, over a long period of time, which helped to increase the accuracy of our model to truly determine the G.O.A.T.

Lastly, our adapted model that does not use the Floyd-Warshall algorithm takes $O(m)$ time, where m is the number of competition records, thus it is not computationally intensive. Hence, it can be used for calculating large data sets in a **relatively short amount of time**.

Limitations

However, there are still certain factors that we could not account for in our model, such as the swimmers' level of sportsmanship, or other character traits that are not revealed through their performance in competitions. Despite this, these factors **cannot be quantified via mathematical means**, thus this is an unavoidable limitation of our model.

Also, our data was not as comprehensive as we would have liked it to be, as we felt that it was also important to consider smaller-scale or regional competitions, which swimmers would also frequently take part in. However, many of these less-publicised competitions **do not publish sufficient data online** that dates back to 1990, and thus, solely including the recent results of these competitions would make it highly unfair for older swimmers who competed more frequently in the past. Also, including the 2 most prominent and international competitions would already allow us to gain data from the most well-known and skilled swimmers, who definitely have the highest likelihood of being known as the G.O.A.T.

4.7. Sensitivity Analysis

We conducted a few variations of sensitivity analysis on our model. Firstly, we varied the normalisation ranges. In our model, the ranges were from [1,1.25] (α_i), [1,1.5] (c_i) and [1,2] (b_i). We varied the upper bound of the normalisation from 1 to 2, while fixing the lower bound at 1, such that the range of normalisation for each of the three factors varies from 1 to 2. The top 3 athletes remained consistent. The graph below in Fig. 4 shows the ratio of top 2 G.O.A.T.-ness scores as we varied the normalisation of each range independently.

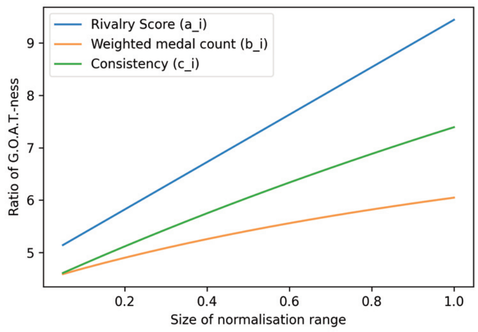

Fig. 4: Variation of ratio of G.O.A.T.-ness score by changing normalisation ranges of different variables

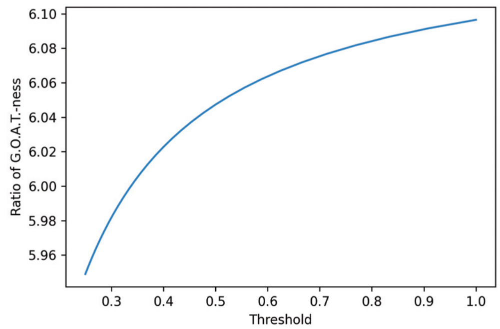

Fig. 5: Variation of ratio of G.O.A.T.-ness score by varying threshold

To test the stability of our results, for each round, we removed 10% of the competition records randomly from the entire dataset, and then found the top 3 greatest swimmers based on their G.O.A.T.-ness score. After executing this process 1000 times, Table 4 summarises the top athletes and their probability of achieving each of the top 3 positions.

Athlete name	1st place	2nd place	3rd place
Michael Phelps	99.9%	0.1%	0%
Melvin Stewart	0.1%	85.1%	4.5%
Kristóf Milák	0%	13.3%	75.8%
Michael Gross	0%	1.1%	8%
Chad Le Clos	0%	0.4%	7.3%

Table 4: Probability of some top athletes achieving each of the top 3 positions, when 10% of data is randomly removed

As seen from the table above, we attain Michael Phelps, Melvin Stewart and Kristóf Milák as our top 3 athletes consistently. Michael Phelps remains as the G.O.A.T. in almost all cases. This shows that our model is robust against changes in the competition data and various parameters used in calculating the G.O.A.T.-ness score.

For the rivalry score, we also varied our threshold for a close margin, which we previously chose to be 0.5s. This threshold is varied between 0.25s to 1.0s. Fig. 5 above shows the variation in ratio of the G.O.A.T.-ness scores of the top two athletes with changing threshold, which shows the ratio of G.O.A.T.-ness scores start to stabilise after the threshold reaches 0.5s. Hence we used 0.5s as our threshold as there would be marginal changes to this ratio after 0.5s.

4.8. Extension of Model to Other Individual Sports (Task 2b)

Due to the differences between one-on-one sports and inanimate sports, it is crucial that we adopt different approaches when attempting to find the G.O.A.T. for the respective sports.

"One-on-one" Sports

“One-on-one” sports are those involving two parties competing directly with one another. Examples of such sports include badminton and tennis (as seen in Task 1). For such sports, we can use a model similar to our model in Task 1, in which the scores of a match between two players are used to create a network with directed edges that allows for comparisons to be drawn between players, and thus a G.O.A.T. can be determined.

Additional factors, such as consistency and medal tally, would have to be added to the model in Task 1, just as we have demonstrated in Task 2. However, the weightage of each factor in the overall function for G_i would have to be different for various sports. For instance, the number of world records should not be factored into G_i, simply because there are no world record timings or scores to surpass.

Furthermore, the component of rivalry can be given greater weightage in one-to-one sports, as the presence of rivalry in one-on-one sports is likely to be higher than sports with inanimate standards.

The nature of these sports allows famous pairs to directly battle against each other, especially in the finals, thus gaining greater attention from the public, and breeding classic rivalries such as Ali and Frasier in boxing.

Sports with an inanimate standard

These sports refer to sports which use rankings, score or time measures to assess athletes' performance; hence, there is a greater emphasis on individual performance. Examples of such sports include golf and swimming (as shown in Task 2a). In such scenarios, we can use a similar model as that used in 2a. As we acknowledge that the abilities of athletes can collectively improve or worsen over time, directed edges can be drawn from players of higher to lower ability based on their differences in score (such as timing or goals), rather than merely using the absolute scores.

Also, edges are only drawn between athletes who had competed in the same competition (i.e. the same round of finals), instead of drawing edges across different competitions, since players from different eras may not be directly competing against each other. We can then derive the ability of athletes from this graph representation as illustrated in Task 2a. Using similar factors in Task 2a, we can find the greatness score, G_i, of each athlete.

5. Task 3

5.1. Additional Considerations

For team sports, there are additional considerations that would have to be made. Team sports often involve a division of labour - each player must undertake a specific role or position, and a set of functions based on the position he or she plays in [21].

For instance, this division is clearly visible in soccer, where players can be the goalkeeper, defenders, midfielders or forwards. Being the main attackers, the forwards are the most likely to score points for the team. However, this would mean that forwards are more likely to be the G.O.A.T., as compared to other roles, should the ranking system depend on the points scored by the player for the team. This suggests that considering the athletes' differing positions could play a key role when assessing performance in team sports.

While it could be possible that the attacker is the most likely to be remembered by the general public and is usually the one who makes headlines, and is thus the most likely to be the G.O.A.T., we still want to recognise exceptional performance in other roles such as goalkeepers.

5.2. Adaptation of Our Previous Model

As such, our measure of "G.O.A.T.-ness" for each athlete in team sports will comprise of 2 components - a "team performance" element and "individual performance" element.

For the **"team performance" element** G_t, the method of calculation will be largely similar to that in Task 2:

$$G_t = d_t \times norm(c_t, 1, 1.5) \times norm(\sqrt{b_t}, 1, 2) \times norm(a_t, 1, 1.25)$$

To obtain the value of d_t, which represents **team ability**, we will construct a network where each team represents a node, and the edges drawn between nodes represent the margins that each team wins another team by, thus using the same concept as the previous tasks. In other words, instead of comparing the abilities of individual athletes, we compare the abilities of entire teams.

The **measure of consistency**, c_t, represents how many times the team has successfully entered the quarter-finals or any prestigious equivalent for each sport. b_t is the **total number of "medal points"** that the team has earned over time, and is again square-rooted and normalised from a range of 1 to 2. Lastly, the **degree of rivalry**, a_t, is determined by the number of times the team has competed in the same finals as another team.

It is important to note that the degree of rivalry will again vary from sport to sport, given the different frequency of competition, and "star power" of individual athletes [22]. For instance, the degree of rivalry for American football is higher than other team sports such as basketball and baseball [23]. Thus, the normalisation range for α_i should be different for various sports too.

To account for the fact that athletes could take part in different teams across their sporting career [24], we can take a weighted average of team abilities, based on the duration the athlete has played in each particular team.

As for the **"individual performance" element**, G_{indiv}, it takes into account the different Key Performance Indicators (KPIs) that athletes with different roles in team sports will have. For instance, using the same example of soccer, a study found that the goalkeepers had a vastly different set of KPIs from the outfield players [25]. Hence, the "individual performance" element for goalkeepers could compare the number of balls successfully defended, while that for attackers could look at the total number of successful passes or goals.

Of course, the KPIs across different sports would vary, but quantifiable indicators or metrics should be adopted. There are many existing KPIs for athletes from a variety of sports, based on whether the sports are net and wall games, invasion games, or striking and fielding games [26], and these can be used to quantify the individual greatness of each athlete. In fact, for baseball, there is even an existing metric, known as the Wins Above Replacement (WAR), that assesses each player's contribution to a team's success, and is based on the quality of his or her batting, base-running, fielding, and pitching [27].

Thus, the overall formula for "G.O.A.T.-ness" is as follows:

$$G_{total} = G_t \times G_{indiv}$$

5.3. Strengths and Limitations of this Adaptation

Strengths

Our model for team sports is largely adapted from the one we used for individual sports, hence it is easier for sports networks, like *Top Sports,* to utilise. Since teams are competing against each other, our method for comparing the relative ability of players using the directed graph was easily adapted for teams.

Also, as we recognised the important fact that players may play for multiple teams in their lifetime, we used a weighted average for the ability of teams to ensure fairness.

Next, it is important to consider the individual achievements or performance of athletes even if they play as a team, as we acknowledge that some roles in team sports are uniquely different from other roles, as we have seen in the soccer example. These players are limited to certain areas of the court, or are only involved in defence. This extends to other sports as well, such as baseball and hockey.

Furthermore, as most teams in team sports from basketball to volleyball have substitutes, it is also necessary to consider the contributions of each player, to judge whether he or she is an important player in the team and how much of the team's success can be attributed to him or her.

Nonetheless, in team sports, a good player is one who is able to work well with the team and bring the team to victory [28]. Hence, a model for team sports would also have to take into account the results that his or her team produces in competitions to determine his or her ranking. To summarise, the team scoring system will effectively select the best teams from the rest, and the individual component is critical in differentiating the best players in the best teams.

Limitations

While our model has many strengths, the effectiveness of our model is still dependent on the amount and type of data present. For other sports besides the ones we studied, the type of data and the specific factors involved will vary slightly, which will mean that minor adjustments will still have to be made to our model, to tailor to each specific sports category's factors.

We also did not consider home advantage of athletes, as it was difficult to quantify the impact of home advantage, where athletes have the tendency to perform better when in their home country.

Lastly, we did not take into account the number of exceptional feats, such as world records, that each athlete achieved in team sports, as it is very difficult to objectively determine what is an "exceptional" feat, and individual athletes do not typically achieve world records in team sports. World records usually only occur in individual sports with inanimate standards.

6. Conclusion

In conclusion, in order to determine the G.O.A.T. for individual and team sports, we crafted a metric for "G.O.A.T.-ness", which took into account the relative abilities of players, their medal tallies, number of times they broke the world record, consistency over the years, as well the intensity of rivalry. These are all important factors to consider, given that there is an objective aspect of the player's abilities, as well as subjective factors that affect the public's recognition of different athletes. In addition, to test our model using real-life data, we obtained competition results from two international swimming competitions over the span of 30 years.

In task 1, our model found that the greatest female single's tennis player of 2018 was Simona Halep, and for task 2, we determined that Michael Phelps was the G.O.A.T. of men's 200-metre butterfly. Having compared our results with real-life rankings and various news articles, we found that the different versions of our model are able to accurately and reliably determine the greatest player in a particular year and the G.O.A.T. for individual sports. From our sensitivity analysis, we have also established that our model is extremely stable, and can cope relatively well with missing data or varying range of normalisations.

As for team sports, we have proposed several changes to the "G.O.A.T.-ness" metric, such as taking into account both team and individual performance, so that athletes with different positions can be fairly judged. Different KPIs and normalisation ranges should also be used for different sports.

The diagram below summarises how our initial model for Task 1 (women's tennis) had been gradually adapted for determining the G.O.A.T. for Task 2 (men's 200-metre butterfly), and subsequently for other individual sports, and finally for team sports.

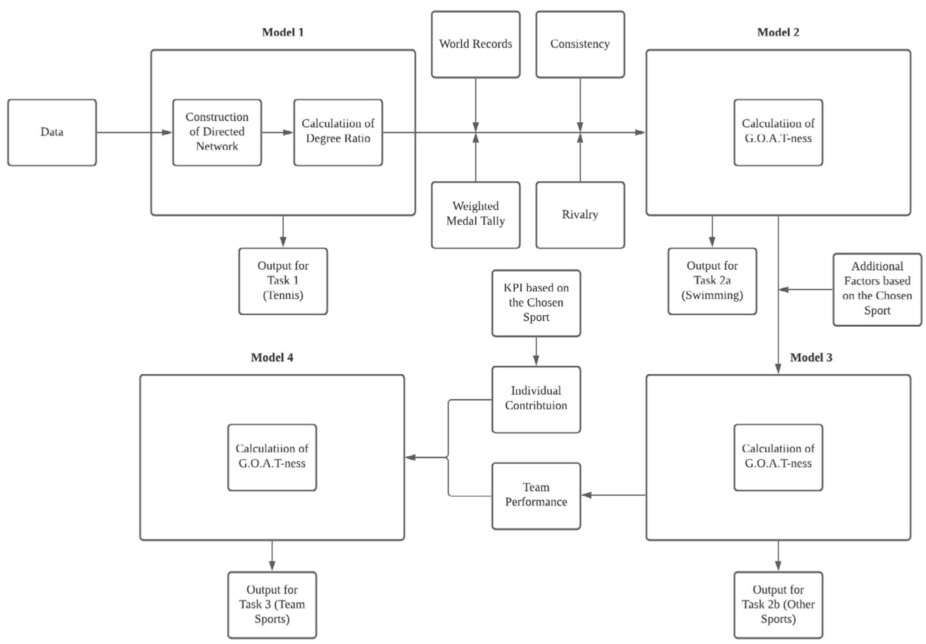

Fig. 6: Summary of how our model was adapted for different types of sports

Lastly, in future studies, with more time, results from a larger range of international competitions can be consolidated, in order to obtain more comprehensive data of the performance of athletes.

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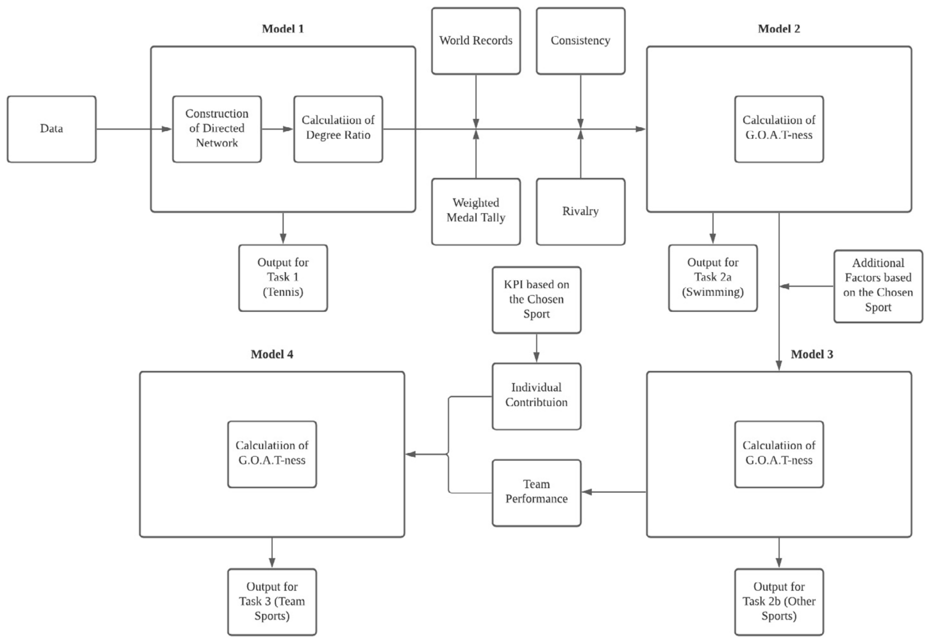

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